Regularity Theory

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1 Interior Regularity

We want to prove the following:

Theorem 1. *Let $k \geq 0$ be an integer and $\mu \in (0,1)$. Let Ω be an open set in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega)$ satisfies*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega$$

for some elliptic operator L with coefficients $a^{ij}, b^i, c \in C^{k,\mu}(\Omega)$ and some $f \in C^{k,\mu}(\Omega)$. Then $u \in C^{k+2,\mu}(\Omega)$ with

$$|u|_{k+2,\mu;\Omega''} \leq C(|u|_{0;\Omega'} + |f|_{k,\mu;\Omega'}) \tag{1}$$

for every $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ for some constant $C = C(n,k,\mu,L,\Omega',\Omega'') \in (0,\infty)$.

Moreover, if $Lu = f$ in Ω for some elliptic operator L with coefficients $a^{ij}, b^i, c \in C^\infty(\Omega)$ and some $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

The proof shall proceed by induction on k.

Step 1: Show that if $a^{ij}, b^i, c, f \in C^{2,\mu}(\Omega)$, then $u \in C^2(\Omega)$ implies that $u \in C^{2,\mu}(\Omega)$. This follows from the existence theory. Let $y \in \Omega$ and $B_\rho(y) \subset\subset \Omega$. Consider solutions v to

$$\begin{aligned} a^{ij} D_{ij} v + b^i D_i v &= f - cu \text{ in } B_\rho(y), \\ v &= u \text{ on } \partial B_\rho(y). \end{aligned} \tag{2}$$

We know that u is a solution to (2) in $C^0(\overline{B_\rho(y)}) \cap C^2(B_\rho(y))$. By the existence theory for solutions to the Dirichlet problem, there exists a solution $v \in C^0(\overline{B_\rho(y)}) \cap C^{2,\mu}(B_\rho(y))$ to (2). Since there is at most one solution to (2) in $C^0(\overline{B_\rho(y)}) \cap C^2(B_\rho(y))$, we conclude that $u = v$. Hence $u \in C^{2,\mu}(B_\rho(y))$.

Step 2: Show that if $a^{ij}, b^i, c, f \in C^{1,\mu}(\Omega)$, then $u \in C^{2,\mu}(\Omega)$ implies that $u \in C^{3,\mu}(\Omega)$. We do this using a difference quotient argument. Let $\Omega''' \subset\subset \Omega'' \subset\subset \Omega' \subset\subset \Omega$. Consider $l \in \{1,2,\ldots,n\}$ and $h > 0$ such that $\operatorname{dist}(\Omega''', \partial\Omega'') < h$ and $\operatorname{dist}(\Omega'', \partial\Omega') < h$. We define the difference quotient operator $\delta_{l,h}$ taking a function $g : \Omega \to \mathbb{R}$ to the function $\delta_{l,h} g : \Omega'' \to \mathbb{R}$ defined by

$$\delta_{l,h} g(x) = \frac{g(x + he_l) - g(x)}{h},$$

where $e_1, e_2, \dots, e_n$ denotes the standard basis for $\mathbb{R}^n$. Note that by applying $\delta_{l,h}$ to both sides of $Lu = f$ in Ω, we get

$$L(\delta_{l,h}u) = \delta_{l,h}f(x) - \delta_{l,h}a^{ij}(x)D_{ij}u(x+he_l) - \delta_{l,h}b^i(x)D_iu(x+he_l) - \delta_{l,h}c(x)u(x+he_l) \text{ on } \Omega''.$$

Now observe that

$$\sup_{x\in\Omega''} |\delta_{l,h}f(x)| = \sup_{x\in\Omega''} \left| \frac{1}{h} \int_0^h D_l f(x+te_l)dt \right| \leq \sup_{\Omega'} |D_l f|$$

and for $x, y \in \Omega''$,

$$|\delta_{l,h}f(x) - \delta_{l,h}f(y)| = \left| \frac{1}{h} \int_0^h (D_l f(x+te_l) - D_l f(y+he_l))dt \right| \leq [D_l f]_\mu |x-y|^\mu.$$

Therefore $|\delta_{l,h}f|_{0,\mu;\Omega''} \leq |D_l f|_{0,\mu;\Omega'}$. Similarly $|\delta_{l,h}u|_{0;\Omega''} \leq |D_l u|_{0;\Omega'}$, $|\delta_{l,h}a^{ij}|_{0,\mu;\Omega''} \leq |D_l a^{ij}|_{0,\mu;\Omega'}$, $|\delta_{l,h}b^i|_{0,\mu;\Omega''} \leq |D_l b^i|_{0,\mu;\Omega'}$, $|\delta_{l,h}c|_{0,\mu;\Omega''} \leq |D_l c|_{0,\mu;\Omega'}$. By the Schauder estimates

$$\begin{aligned}
|\delta_{l,h}u|_{2,\mu;\Omega'''} \leq\, & C(|\delta_{l,h}u|_{0;\Omega''} + |\delta_{l,h}u|_{0,\mu;\Omega''} + |\delta_{l,h}a^{ij}|_{0,\mu;\Omega''}|D_{ij}u(x+he_l)|_{0,\mu;\Omega''} \\
& + |\delta_{l,h}b^i|_{0,\mu;\Omega''}|D_iu(x+he_l)|_{0,\mu;\Omega''} + |\delta_{l,h}c|_{0,\mu;\Omega''}|u(x+he_l)|_{0,\mu;\Omega''}) \\
\leq\, & C(|D_l u|_{0;\Omega''} + |D_l u|_{0,\mu;\Omega''} + |D_l a^{ij}|_{0,\mu;\Omega'}|D^2u|_{0,\mu;\Omega'} \\
& + |D_l b^i|_{0,\mu;\Omega'}|Du|_{0,\mu;\Omega'} + |D_l c|_{0,\mu;\Omega'}|u|_{0,\mu;\Omega'})
\end{aligned}$$

for some constant $C = C(n, \mu, L, \Omega''', \Omega'') \in (0, \infty)$; in other words,

$$|\delta_{l,h}u|_{2,\mu;\Omega'''} \leq C|u|_{2,\mu;\Omega'}$$

for some constant $C = C(n, \mu, L, \Omega''', \Omega'') \in (0, \infty)$. By Arzela-Ascoli, there is a sequence $h_j \downarrow 0$ such that $\delta_{l,h_j}u$ converges in $C^2(\overline{\Omega'''})$. But since u is continuously differentiable on Ω, $\delta_{l,h_j}u \to D_l u$ uniformly on Ω'''. Therefore $\delta_{l,h_j}u \to D_l u$ in $C^2(\overline{\Omega'''})$ and $D_l u \in C^{2,\mu}(\overline{\Omega'''})$. Note that Ω''' is an arbitrary open set compactly contained in Ω. Consequently $u \in C^{3,\mu}(\Omega)$.

Step 3: Show that for $k \geq 2$, if $a^{ij}, b^i, c, f \in C^{k,\mu}(\Omega)$, then $u \in C^{k+1,\mu}(\Omega)$ implies that $u \in C^{k+2,\mu}(\Omega)$. Let $|\alpha| = k-1$ and observe that by applying D^α to both sides of $Lu = f$ in Ω,

$$L(D^\alpha u) = D^\alpha f - \sum_{\beta<\alpha} \frac{\alpha!}{\beta!(\beta-\alpha)!}(D^{\beta-\alpha}a^{ij}D^\beta D_{ij}u + D^{\beta-\alpha}b^iD^\beta D_iu + D^{\beta-\alpha}cD^\beta u) \text{ in } \Omega,$$

where for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, $\beta < \alpha$ means that $\beta_i \leq \alpha_i$ for all $i = 1, 2, \dots, n$ and $\beta_i < \alpha_i$ for some $i \in \{1, 2, \dots, n\}$. Since $a^{ij}, b^i, c, f \in C^{k,\mu}(\Omega)$, $D^{\alpha-\beta}a^{ij}, D^{\alpha-\beta}b^i, D^{\alpha-\beta}c \in C^{1,\mu}(\Omega)$ whenever $\beta < \alpha$ and $D^\alpha f \in C^{1,\mu}(\Omega)$. Therefore $L(D^\alpha u) \in C^{1,\mu}(\Omega)$. By Step 2, $D^\alpha u \in C^{3,\mu}(\Omega)$. Consequently $u \in C^{k+2,\mu}(\Omega)$.

Step 4: (1) follows inductively from the standard interior Schauder estimate. We shall leave this as an exercise to the reader.

Note that the above difference quotient argument is a very general type of argument and the same sort of argument will apply in the cases of global regularity and Sobolev solutions to elliptic equations in divergence form. The key ingredients is a Schauder estimate and having difference quotients the necessary properties for the difference quotient argument to work. In particular, we used:

(a) an interior Schauder estimate for solutions in $C^{2,\mu}$,

(b) the difference quotient operator $\delta_{l,h}$ such that

  (i) $\delta_{l,h}$ is well-defined,

  (ii) the norm of difference quotients $\delta_{l,h} g$ are uniformly bounded by the norm of the derivative $D_l g$ for every function g, i.e. $|\delta_{l,h} g|_{0,\mu;\Omega''} \leq |D_l g|_{0,\mu;\Omega'}$ whenever $g \in C^{1,\mu}(\Omega)$ and $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ with $\operatorname{dist}(\Omega'', \partial\Omega') > h$, and

  (iii) the convergence of the difference quotients $\delta_{l,h} g$ to the derivative $D_l g$, i.e. if $g \in C^{0,\mu}(\Omega)$ satisfies $\sup_{0<|h|<h_0} |\delta_{l,h} g|_{0,\mu;\Omega'''} < \infty$ for some $h_0 > 0$ and $\Omega''' \subset\subset \Omega$, then $\delta_{l,h} g \to D_l g$ uniformly on Ω''' and thus $D_l g \in C^{0,\mu}(\Omega''')$.

2 Global Regularity

We want to prove the following:

Theorem 2. *Let $k \geq 0$ be an integer and $\mu \in (0,1)$. Let Ω be a bounded $C^{k,\mu}$ domain in $\mathbb{R}^n$. Suppose $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{aligned} Lu = a^{ij} D_{ij} u + b^i D_i u + cu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

for some elliptic operator L with coefficients $a^{ij}, b^i, c \in C^{k,\mu}(\overline{\Omega})$, some $f \in C^{k,\mu}(\overline{\Omega})$, and some $\varphi \in C^{k+2,\mu}(\overline{\Omega})$. Then $u \in C^{k+2,\mu}(\overline{\Omega})$ with

$$|u|_{k+2,\mu;\Omega} \leq C(|u|_{0;\Omega} + |f|_{k,\mu;\Omega} + |\varphi|_{k+2,\mu;\Omega})$$

for some constant $C = C(n,k,\mu,L,\Omega) \in (0,\infty)$.

Moreover, if $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$ for a smooth domain Ω, some elliptic operator L with coefficients $a^{ij}, b^i, c \in C^\infty(\overline{\Omega})$ and some functions $f, \varphi \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.

Note that we already know that $u \in C^{k+2,\mu}(\Omega)$, so the issue is establishing $u \in C^{k+2,\mu}$ near and up to the boundary of Ω. Our approach will be to locally establish $u \in C^{k+2,\mu}$ near and up to the boundary of Ω. Consequently, our approach actually proves the following, though we will focus on the proof of Theorem 2.

Theorem 3. *Let $k \geq 2$ be an integer and $\mu \in (0,1)$. Let Ω be an open set in $\mathbb{R}^n$ and let T be a $C^{k,\mu}$ portion of $\partial\Omega$ (i.e. for every $y \in T$, there is a ball $B_\rho(y)$ and straightening diffeomorphism $\Phi : B_\rho(y) \to \Phi(B_\rho(y)) \subseteq \mathbb{R}^n$ such that $\partial\Omega \cap B_\rho(y) = T \cap B_\rho(y)$, $\Phi(\Omega \cap B_\rho(y)) \subseteq \mathbb{R}^n_+$ and $\Phi(T \cap B_\rho(y)) \subseteq \mathbb{R}^{n-1} \times \{0\}$). Suppose $u \in C^2(\Omega \cup T)$ satisfies*

$$\begin{aligned} Lu = a^{ij} D_{ij} u + b^i D_i u + cu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } T, \end{aligned}$$

for some elliptic operator L with coefficients $a^{ij}, b^i, c \in C^{k,\mu}(\Omega \cup T)$, some $f \in C^{k,\mu}(\Omega \cup T)$, and some $\varphi \in C^{k+2,\mu}(\Omega \cup T)$. Then $u \in C^{k+2,\mu}(\Omega \cup T)$.

Moreover, if $Lu = f$ in Ω and $u = \varphi$ on T for an open set Ω, smooth portion T of $\partial\Omega$, some elliptic operator L with coefficients $a^{ij}, b^i, c \in C^\infty(\Omega \cup T)$ and some functions $f, \varphi \in C^\infty(\Omega \cup T)$, then $u \in C^\infty(\Omega \cup T)$.

The proof of Theorem 2 proceeds by induction on k and is similar to the proof of Theorem 1.

Step 1': Show that if Ω is a $C^{2,\mu}$ domain, $a^{ij}, b^i, c, f \in C^{0,\mu}(\overline{\Omega})$, and $\varphi \in C^{2,\mu}(\overline{\Omega})$, then $u \in C^2(\overline{\Omega})$ implies that $u \in C^{2,\mu}(\overline{\Omega})$. In particular, we will show that if $y \in \partial\Omega$ and T is a neighborhood of y in $\partial\Omega$ that is $C^{2,\mu}$, then u is in $C^{2,\mu}$ up to the boundary of Ω near y.

Choose neighborhood T' of y in $\partial\Omega$ with $T' \subset\subset T$ and a small $C^{2,\mu}$ domain $D \subset \Omega$ such that $T' \subset \partial D$ and D is small enough that there exist at most one solution u' to the Dirichlet problem

$$\begin{aligned} Lu' &= f' \text{ in } D, \\ u' &= \varphi' \text{ on } \partial D \end{aligned} \tag{3}$$

for each $f' \in C^0(\overline{D})$ and $\varphi' \in C^0(\partial D)$. By the Fredholm Alternative, there exists a unique solution $u' \in C^{2,\mu}(\overline{D})$ to (3) for each $f' \in C^{0,\mu}(\overline{D})$ and $\varphi' \in C^{2,\mu}(\overline{D})$.

Pick a domain D' with $D \subset\subset D'$ and a small ball $B = B_\rho(y)$ with $B \subset\subset\subset D'$, $B \cap \partial\Omega = B \cap T'$, and $B \cap \Omega = B \cap D$. Consider $\chi = u|_{\partial D} \in C^0(\partial D) \cap C^{2,\mu}(T')$ and extend χ to a function in $C^0(D') \cap C^{2,\mu}(\overline{B})$. Use convolution to approximate χ by $\chi_k \in C^\infty(D')$ such that $\chi_k \to \chi$ uniformly on $\overline{D}$, $|\chi_k|_{0;D} \leq C|\chi|_{0;D}$, and $|\chi_k|_{2,\mu;B} \leq C|\chi|_{2,\mu;B}$ for some constant $C \in (0,\infty)$ independent of k. Let $u_k \in C^{2,\mu}(\overline{D})$ be the solution to

$$\begin{aligned} Lu_k &= f \text{ in } D, \\ u_k &= \chi_k \text{ on } \partial D. \end{aligned}$$

We claim that by the maximum principle, interior Schauder estimates, and Schauder estimates at the boundary near y, u_k converges to some function v uniformly on $\overline{D}$, in C^2 on compact subsets of D, and in $C^2(\overline{\Omega \cap B_{\rho/2}(y)})$ such that $v \in C^0(\overline{D}) \cap C^{2,\mu}(D) \cap C^{2,\mu}(\overline{\Omega \cap B_{\rho/2}(y)})$ and $w = v$ solves

$$\begin{aligned} Lw &= f \text{ in } D, \\ w &= \chi \text{ on } \partial D. \end{aligned} \tag{4}$$

Since $w = u$ also solves (4), it follows that $u = v$ and thus $u \in C^{2\,mu}(\overline{\Omega} \cap B_{\rho/2}(y))$, as required. To establish the convergence of u_k, first observe that by the maximum principle,

$$|u_k - u_l|_{0;D} \leq |\chi_k - \chi_l|_{0;\partial D},$$

so u_k is Cauchy in $C^0(\overline{D})$ and u_k converges to some function v in $C^0(\overline{D})$. By the interior Schauder estimates,

$$\begin{aligned} |u_k|_{2,\mu;D''} &\leq C(|\chi_k|_{0;\partial D} + |f|_{0,\mu;D}) \\ &\leq C(|\chi|_{0;D} + |f|_{0,\mu;\Omega}) \end{aligned}$$

for all $D'' \subset\subset D$ for some constant $C \in (0,\infty)$ independent of k, so after passing to a subsequence $u_k \to v$ in C^2 on compact subsets of D and $v \in C^{2,\mu}(D)$. By the Schauder estimates near the boundary,

$$\begin{aligned} |u_k|_{2,\mu;\Omega\cap B_{\rho/2}(y)} &\leq C(|\chi|_{0;\partial D} + |f|_{0,\mu;D} + |\chi_k|_{2,\mu;\Omega\cap B}) \\ &\leq C(|\chi|_{0;\partial D} + 1 + |f|_{0,\mu;D} + |\chi|_{2,\mu;B}), \end{aligned}$$

for some constant $C \in (0,\infty)$ independent of k, so after passing to a subsequence $u_k \to v$ in $C^2(\overline{\Omega} \cap \overline{B_{\rho/2}(y)})$ and $v \in C^{2,\mu}(\overline{\Omega} \cap \overline{B_{\rho/2}(y)})$.

Step 2': Show that if Ω is a $C^{1,\mu}$ domain, $a^{ij}, b^i, c, f \in C^{1,\mu}(\overline{\Omega})$, and $\varphi \in C^{3,\mu}(\overline{\Omega})$, then $u \in C^{2,\mu}(\overline{\Omega})$ implies that $u \in C^{3,\mu}(\overline{\Omega})$. Let $y \in \partial\Omega$. First we use a $C^{3,\mu}$ straightening diffeomorphism $\Phi : B_\rho(y) \to \Phi(B_\rho(y)) \subseteq \mathbb{R}^n$ such that

$$\Phi(\Omega \cap B_\rho(y)) \subseteq \mathbb{R}^n_+, \quad \Phi(\partial\Omega \cap B_\rho(y)) \subseteq \mathbb{R}^{n-1} \times \{0\}.$$

Assume without loss of generality that $B_1(0) \subset\subset \Phi(B_\rho(y))$. Let $\tilde{u} = u \circ \Phi^{-1}$, $\tilde{f} = f \circ \Phi^{-1}$, and $\tilde{\varphi} = \varphi \circ \Phi^{-1}$ on $B_1(0)$. Recall that $\tilde{u}$ satisfies

$$\begin{aligned} \tilde{L}\tilde{u} = \tilde{a}^{ij} D_{ij}\tilde{u} + \tilde{b}^i D_i \tilde{u} + \tilde{c}\tilde{u} = \tilde{f} \text{ in } B_1^+(0), \\ \tilde{u} = \tilde{\varphi} \text{ on } B_1^{n-1}(0) \times \{0\}, \end{aligned}$$

where $\tilde{L}$ is an elliptic operator with some coefficients $\tilde{a}^{ij}, \tilde{b}^i, \tilde{c} \in C^{1,\mu}(\overline{B_1^+(0)})$. Now using the Schauder estimates on half-balls instead of the interior Schauder estimates we can use the same difference quotient from Step 2 for the interior estimates to get $D_l\tilde{u} \in C^{2,\mu}(\overline{B^+_{1/4}(0)})$ for all $l = 1, 2, \ldots, n$. However, the difference quotient $\delta_{n,h}$ in the direction orthogonal to $B_1^{n-1}(0) \times \{0\}$ is not well-defined on $B^+_{1/2}(0)$, so we cannot use difference quotients to show that $D_n\tilde{u} \in C^{2,\mu}(\overline{B^+_{1/4}(0)})$. Thus we observe the following. Since $D_i\tilde{u} \in C^{2,\mu}(\overline{B^+_{1/4}(0)})$ for $i = 1, 2, \ldots, n-1$, then

$$\begin{aligned} &D_{ij}\tilde{u} \in C^{1,\mu}(\overline{B^+_{1/4}(0)}) \text{ for } i, j = 1, 2, \ldots, n-1, \\ &D_{in}\tilde{u} \in C^{1,\mu}(\overline{B^+_{1/4}(0)}) \text{ for } i = 1, 2, \ldots, n-1. \end{aligned}$$

Thus it remains to show that $D_{nn}\tilde{u} \in C^{1,\mu}(\overline{B^+_{1/4}(0)})$. Since $\tilde{L}\tilde{u} = \tilde{f}$ in $B_1^+(0)$ and $\tilde{L}$ is elliptic,

$$D_{nn}\tilde{u} = \frac{1}{\tilde{a}^{nn}} \left(\tilde{f} - \sum_{i,j \neq (n,n)} \tilde{a}^{ij} D_{ij}\tilde{u} - \sum_{i=1}^n \tilde{b}^i D_i\tilde{u} - \tilde{c}\tilde{u} \right) \in C^{1,\mu}(\overline{B^+_{1/4}(0)}).$$

Step 3': Show that for $k \geq 4$, if Ω is a $C^{k,\mu}$ domain, $a^{ij}, b^i, c, f \in C^{k,\mu}(\overline{\Omega})$, $\varphi \in C^{k+2,\mu}(\overline{\Omega})$, then $u \in C^{k+1,\mu}(\overline{\Omega})$ implies that $u \in C^{k+2,\mu}(\Omega)$. This follows from Step 2' exactly like with interior regularity.

Step 4': (1) follows inductively from the standard global Schauder estimate and is nearly identical to Step 4 for interior regularity.

References: Gilbarg and Trudinger, Section 6.4